NON-SINGLY-GENERATED CLOSED IDEALS IN GROUP ALGEBRAS

BY

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ABSTRACT

We investigate the problem whether every closed ideal in the group algebras $A(G)$ is generated by a single function. For the algebras $A_r(R^n)$, $n \geqq 6$, we give a negative solution. We also obtain some negative results for general locally compact abelian groups.

Introduction. For a locally compact abelian group G with dual Γ, we denote by $A(G)$ the set of all Fourier transforms of functions in $L^1(\Gamma)$. That is:

$$f \in A(G) \Leftrightarrow f(g) = \int_\Gamma (g,\gamma) F(\gamma) d\gamma,$$

$$g \in G, \ \gamma \in \Gamma, \ F \in L^1(\Gamma);$$

with the norm

$$\|f\| = \int_\Gamma |F(\gamma)| d\gamma$$

$A(G)$ is a commutative Banach algebra and G is its maximal ideal space. $A_r(G)$ denotes the algebra of real functions in $A(G)$.

It is not known whether every closed ideal in $A(G)$ is generated by a single function. In this paper we discuss this problem, and prove some results in the negative direction.

In Section 1 we investigate the relation of this problem to spectral synthesis.

In Section 2 we consider the algebras $A(R^n)$ for $n \geqq 6$, and show the existence of closed self adjoint ideals which are not generated by any real function. This

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implies that in the algebras $A_r(R^n)$, $n \geqq 6$, there exist closed ideals which are not singly generated.

Section 3 is devoted to the study of the problem in general algebras $A(G)$, for non discrete locally compact abelian groups G. We show, that in every such algebra, there exist two real functions f_1, f_2, and a compact set $E \subset G$, such that the closed ideal generated by f_1 and f_2 in $A(G)$ is not generated by any function f which identifies every pair of points in E that is identified by both f_1 and f_2. This implies that $A(G)$ contains a closed subalgebra with closed ideals which are not singly generated.

1. **Relation to spectral synthesis.** For every f in $A(G)$ we denote by $Z(f)$ the set of its zeros. If I is a closed ideal in $A(G)$ we shall denote by $Z(I)$ the set of common zeros of all functions in I. That is,

$$Z(I) = \bigcap_{f \in I} Z(f).$$

We shall denote by $I(f)$ the closed ideal generated by f. Clearly,

$$Z(I(f)) = Z(f). \tag{1.1}$$

This shows, that for a closed singly generated ideal I, $Z(I)$ is a closed G_δ set. For a closed $E \subset G$ we denote by $K(E)$ the ideal of all functions in $A(G)$ which vanish on E. It is well known that the algebras $A(G)$ are regular, that is, for ever closed set $E \subset G$ we have:

$$Z(K(E)) = E. \tag{1.2}$$

From (1.1) and (1.2) it follows that if E is a closed set which is not a G_δ, $K(E)$ is not singly generated. Recall that a closed set $E \subset G$ is a set of spectral synthesis (an S set) if for every closed ideal I such that $Z(I) = E$ we have $I = K(E)$. The regularity of $A(G)$ implies that for every closed G_δ set $E \subset G$ there exists an f in $A(G)$ such that $Z(f) = E$ [7, pp. 419]. From this we infer that every closed ideal I such that $Z(I)$ is G_δ, and of spectral synthesis is singly generated. Hence, if G is discrete, every closed ideal in $A(G)$ is singly generated, since in this case every subset of G is a S-set [4, pp. 159]. If every closed subset of G is a G_δ, in particular if G is metriziable, it follows from the above discussion, that the problem of singly generated closed ideals in $A(G)$ involves the consideration of sets which are not of spectral synthesis. It was proved by P. Malliavin [2] that every non-discrete locally compact abelian group G contains a closed subset which is not of spectral synthesis.

For a closed set $E \subset G$ which is not an S-set it is natural to ask whether $K(E)$ can be singly generated. We show that this can be the case. The well known example of L. Schwartz [5] shows that the unit sphere S_n in Euclidean n-space R^n is not an S-set for $n \geqq 3$. However, $K(S_n)$ is singly generated. This can be seen as follows: Let $\mathscr{R}_n$ be the algebra of radial functions in $A(R^n)$. Consider the closed ideal $K_{\mathscr{R}} = K(S_n) \cap \mathscr{R}_n$ in $\mathscr{R}_n$. It is proved in [6] that $K_{\mathscr{R}}$ generates $K(S_n)$ in $A(R^n)$. It is easy to see that $K_{\mathscr{R}}$ is generated in $\mathscr{R}_n$ by any f in $\mathscr{R}_n$ such thar $Z(f) = S_n$ and $f(x) = \sum_{k=1}^{n} x_k^{\,2} - 1$ for x in a neighborhood of S_n. Hence $K(S_n)$ is also generated by the same f.

2. **The algebras** $A(R^n)$. We consider the algebras $A(R^n)$ for $n \geqq 6$. In what follows p, q will denote positive integers such that $p + q = n$. Let $S_{p,q} = S_p \times S_q$. This section is devoted to the investigation of the ideal $K_{p,q} = K(S_{p,q})$. We consider R^n as $R^p \times R^q$, and use the notations: $x = (u,v)$, $x \in R^n$, $u \in R^p$, $v \in R^q$.

$$r_1 = \| u \| = \left(\sum_{k=1}^{p} u_k^{\,2} \right)^{1/2}, \; r_2 = \| v \| = \left(\sum_{k=1}^{q} v_k^{\,2} \right)^{1/2}.$$

We shall denote by $\mathbf{P}^2$ the positive quadrant in the plane i.e.,

$$\mathbf{P}^2 = \{x \in R^2 \colon \; x_j \geqq 0, \; j = 1,2.\}.$$

Every point in R^n can be written in the form $(r_1 s_1, r_2 s_2)$ where $(r_1, r_2) \in \mathbf{P}^2$ and $(s_1, s_2) \in S_{p,q}$. We shall also denote by $\mathscr{R}_{p,q}$ the algebra of bi-radial functions in $A(R^n)$ with respect to the pair (p,q). That is, $\mathscr{R}_{p,q}$ consists of those f in $A(R^n)$ such that $f(u_1, v_1) = f(u_2, v_2)$ whenever $\| u_1 \| = \| u_2 \|$ and $\| v_1 \| = \| v_2 \|$. Every f in $\mathscr{R}_{p,q}$ will be identified with a function defined on $\mathbf{P}^2$. By abuse of language we shall sometimes write $f(u,v) = f(r_1, r_2)$ for f in $\mathscr{R}_{p,q}$. It is known [6] that for $p \geqq 3$, $q \geqq 3$, every $f \in \mathscr{R}_{p,q}$ has continuous first order partial derivatives for $r_j > 0$, $j = 1,2$, and the mappings $\delta_j'(1,1) \colon f \to \partial f / \partial r_j(1,1)$, $j = 1,2$, $f \in \mathscr{R}_{p,q}$, are bounded linear functionals on $\mathscr{R}_{p,q}$. Let μ_k be the homogeneously distributed measure on S_k with total mass 1, and $\mu_{p,q} = \mu_p \times \mu_q$ the product measure on $S_{p,q}$. For every continuous function f on R^n, we define its bi-radial mean by:

$$M_{p,q}(f)(r_1, r_2) = \int_{S_{p,q}} f(r_1 s_1, r_2 s_2) \, d\mu_{p,q}(s_1, s_2), \qquad (r_1, r_2) \in \mathbf{P}^2. \tag{2.1}$$

$M_{p,q}$ restricted to $A(R^n)$ is a bounded linear operator onto $\mathscr{R}_{p,q}$. The following properties of $M_{p,q}$ are easily verified:

$$M_{p,q} f = f \quad \text{for } f \text{ in } \mathscr{R}_{p,q}; \tag{2.2}$$

(2.3) $$M_{p,q}(fg) = fM_{p,q}(g) \quad \text{for } f \in \mathscr{R}_{p,q}, \qquad g \in A(R^n);$$

(2.4) $$|M_{p,q}(fg)(r_1, r_2)| \leqq \|g\| M_{p,q}(|f|)(r_1, r_2) \qquad \text{for } (r_1, r_2) \in \mathbf{P}^2, f, g \text{ in } A(R^n)$$

Finally we define two bounded linear functionals Δ_k, $k = 1, 2$ on $A(R^n)$:

(2.5) $$\langle \Delta_k, f \rangle = \langle \delta_k', M_{p,q} f \rangle, \qquad k = 1, 2.$$

THEOREM 2.1. *For $p \geqq 3$, $q \geqq 3$, the ideal $K_{p,q}$ is not generated by any f in $A_r(R^n)$.*

Proof. Let $f \in A_r(R^n)$, $Z(f) = S_{p,q}$. We shall show that $I(f) \neq K_{p,q}$. The complement of $S_{p,q}$ in R^n is connected, hence f has fixed sign outside $S_{p,q}$, and we may assume that

(2.6) $$f(x) > 0 \quad \text{for } x \notin S_{p,q}, \; Z(f) = S_{p,q}.$$

By (2.1) and (2.6) we have:

(2.7) $$M_{p,q}(f)(1, 1) = 0;$$

(2.8) $$M_{p,q}(f)(r_1, r_2) > 0, \quad (r_1, r_2) \neq (1.1).$$

Hence $M_{p,q}(f)$ has a local minimum at (1,1), and we obtain:

(2.9) $$\langle \Delta_1, f \rangle = 0.$$

From (2.4) and (2.6) it follows that for every g in $A(R^n)$,

$$|M_{p,q}(fg)(r_1, r_2)| \leqq \|g\| M_{p,q}(f)(r_1, r_2)| \quad \text{for all } (r_1, r_2) \in \mathbf{P}^2.$$

This together with (2.7) and (2.9) implies:

(2.10) $$\langle \Delta_1, fg \rangle = 0.$$

Since (2.10) holds for every g in $A(R^n)$, we infer that Δ_1 annihilates $I(f)$. But Δ_1 does not annihilate $K_{p,q}$ since for h in $\mathscr{R}_{p,q}$ which is equal to $r_1 - 1$ in a neighborhood of $S_{p,q}$ we have $\langle \Delta_1, h \rangle = 1$. This completes the proof.

COROLLARY 1. *In the algebras $A(R^n)$, $n \geqq 6$, there exist self adjoint closed ideals which are not generated by any function in $A_r(R^n)$.*

COROLLARY 2. *The algebras $A_r(R^n)$, $n \geqq 6$, contain closed ideals which are not singly generated.*

REMARK. It is proved in [4, 181–183] that for every non-discrete locally compact abelian group G, $A(G)$ contains closed ideals which are not self adjoint.

Clearly, such ideals are not generated by any f in $A_r(G)$. Hence the main point in Corollary 1 is that for $G = R^n$, $n \geqq 6$, $A(G)$ contains self adjoint closed ideals which are not generated by any real function.

In Section 1 we have seen that the ideals $K(S_n)$ are generated by a radial function. We show now that the corresponding result does not hold for $K_{p,q}$.

THEOREM 2.2. *For* $p \geqq 3$, $q \geqq 3$, *the ideal* $K_{p,q}$ *is not generated by any function in* $\mathscr{R}_{p,q}$.

Proof. Let $f \in \mathscr{R}_{p,q} \cap K_{p,q}$; hence,

$$f(1,1) = 0. \tag{2.11}$$

Let c_1, c_2, be constants, $|c_1| + |c_2| > 0$, such that

$$c_1 \frac{\partial f}{\partial r_1}(1,1) + c_2 \frac{\partial f}{\partial r_2}(1,1) = 0. \tag{2.12}$$

Let Δ_1, Δ_2 be as in (2.5), we define:

$$\Delta = c_1\Delta_1 + c_2\Delta_2 .$$

By (2.11), (2.12) and (2.3) we have:

$$\langle \Delta, fg \rangle = 0 \tag{2.13}$$

for every g in $A(R^n)$, hence Δ annihilates $I(f)$. Taking h in $\mathscr{R}_{p,q}$ which coincides with $\bar{c}_1(r_1 - 1) + \bar{c}_2(r_2 - 1)$ in a neighborhood of $S_{p,q}$ we have $h \in K_{p,q}$ but $\langle \Delta, h \rangle = |c_1|^2 + |c_2|^2 \neq 0$; hence Δ does not annihilate $K_{p,q}$. This shows that $I(f) \neq K_{p,q}$, and the theorem is proved.

COROLLARY. *The algebras* $\mathscr{R}_{p,q}$ $p \geqq 3$, $q \geqq 3$, *contain closed ideals which are not singly generated.*

REMARK. All the preceding results are true for the algebras $A(T^n)$, $n \geqq 6$. (T as usual, denotes the circle group). This follows from the well known local isomorphism between $A(R^n)$ and $A(T^n)$, if we replace the spheres S_p and S_q by sufficiently small spheres in T^p and T^q respectively.

The results of this section strongly support the conjecture that the ideals $K_{p,q}$, $p \geqq 3$, $q \geqq 3$, are not singly generated in $A(R^n)$. However, one can show as in Section 1, that $K_{p,q}$ is generated by two functions in $\mathscr{R}_{p,q}$.

3. **General Groups.** Let G be a locally compact abelian group, and f_1, f_2, functions in $A_r(G)$. We denote as usual by $C(R^2)$ the set of all continuous com-

plex functions on R^2. For Φ in $C(R^2)$ we shall denote by $\Phi(f_1,f_2)$ the continuous function on G whose value at every $g \in G$ is $\Phi(f_1(g),f_2(g))$. For every subset $E \subset G$ we denote by $[f_1,f_2]_E$ the set of all f in $A(G)$ for which there exist some Φ in $C(R^2)$ such that $f(g) = \Phi(f_1,f_2)(g)$ for every $g \in E$. It is easy to see that $[f_1,f_2]_E$ is the set of all functions in $A(G)$, which identify every pair of points in E, which is identified by both f_1 and f_2. Clearly $[f_1,f_2]_E$ is a closed subalgebra of $A(G)$.

THEOREM 3.1. *Let G be a non-discrete locally compact abelian group. There exist two functions f_1 and f_2 in $A_r(G)$ and a compact subset $E \subset G$, such that the closed ideal generated by f_1 and f_2 in $A(G)$, is not generated by any $f \in [f_1,f_2]_E$.*

In the proof of Theorem 3.1 we shall use the methods introduced by P. Malliavin in [3]. We shall need the following:

LEMMA. *Let G be as in Theorem 3.1. There exist two functions f_1 and f_2 in $A_r(G)$ and a positive, non-trivial, regular Borel measure ν on G, with compact support, such that,*

$$\| \exp[-i(uf_1 + vf_2)]\nu \|_{A^*(G)} \leqq (u^2 + v^2)^{-2} \tag{3.1}$$

for all real u and v, $(u,v) \neq (0,0)$.

This lemma is proved in [3] for non-discrete compact groups G, with ν as the Haar measure of G. The general case follows from the compact one by using the fact that for every non-discrete locally compact abelian group G, there exist a non discrete compact abelian group H, such that $A(G)$ and $A(H)$ are locally isomorphic. This follows from [4, pp. 41, 55, 56].

Proof of Theorem 3.1. Consider f_1,f_2 and ν which satisfy (3.1). We shall denote by $I(f_1,f_2)$ the closed ideal generated by f_1 and f_2 in $A(G)$. Let E be the support of ν. We shall show that for $f \in I(f_1,f_2) \cap [f_1,f_2]_E$ we have, $I(f) \neq I(f_1,f_2)$. For every h in $A(G)$, let $d\mu_h$ be the image of the measure $h\,d\nu$ by the continuous map:

$$g \to (f_1(g),\, f_2(g)), \quad g \in E;$$

$d\mu_h$ is a measure on R^2 with compact support, and for every $\Phi \in C(R^2)$ we have:

$$\int_{R^2} \Phi(x,y)d\mu_h(x,y) = \int_G \Phi(f_1(g),f_2(g))\,h(g)\,d\nu(g). \tag{3.2}$$

Hence the Fourier transform of $d\mu_h$ is given by:

$$(3.3) \qquad \hat{\mu}_h(u,v) = \int_G \exp[-i(uf_1(g) + vf_2(g)]\, h(g)\, dv(g).$$

From (3.1) and (3.3) we infer that $|\hat{\mu}(u,v)| \leqq \|h\|(u^2+v^2)^{-2}$, $(u,v) \neq (0,0)$. Therefore by the inversion theorem

$$d\mu_h(x,y) = m_h(x,y)dxdy$$

where m_h is a differentiable function on R^2 with compact support. Taking h in $A(G)$ such that $h(g) = 1$ for $g \in E$, it follows from (3.3) that $\hat{m}_1(0,0) = v(E) \neq 0$. Hence m_1 is not identically zero, and by replacing f_1 and f_2 by $f_1 + ak$ and $f_2 + bk$, where a, b are real constants, and k a function in $A_r(G)$ which is equal to 1 on E, (3.1) is not altered, and we may assume that:

$$(3.4) \qquad m_1(0,0) \neq 0.$$

For every h in $A(G)$ we have:

$$(3.5) \qquad (D_1 m_h)(0,0) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(iu)\langle\exp[i(uf_1+vf_2)]v,h\rangle\, dudv;$$

$$(3.6) \qquad (D_2 m_h)(0,0) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(iv)\langle\exp[i(uf_1+vf_2)]v,h\rangle\, dudv.$$

(Here D_1 and D_2 denote differentiation with respect to x and y respectively).

From (3.1), (3.5), and (3.6) it follows that the mappings:

$$\omega_k\colon h \to (D_k m_h)(0,0), \quad k = 1,2, \quad h \in A(G),$$

are bounded linear functionals on $A(G)$. Let $f \in [f_1, f_2]_E$. It is easily verified that for every $\Phi \in C(R^2)$ such that $\Phi(f_1, f_2) = f$ on E, and $h \in A(G)$ we have:

$$(3.7) \qquad m_{fh} = \Phi m_h.$$

By taking $h \in A(G)$ such that $h = 1$ on E, we infer from (3.4) and (3.7) that Φ is differentiable at $(0,0)$. Assume now in addition that $f \in I(f_1, f_2)$. We shall show that

$$(3.8) \qquad \Phi(0,0) = 0.$$

From (3.5) we see that the support[2] of ω_1 is contained in E. From (3.7) it follows that for every $h \in A(G)$ we have,

$$\langle\omega_1, (f_1{}^2 + f_2{}^2)h\rangle = 0.$$

[2] For the definition of support of a bounded linear functional on $A(G)$, and related properties, we refer the reader to [1, Chap. VIII].

Thus ω_1 annihilates $I(f_1{}^2 + f_2{}^2)$, hence its support lies in $Z(f_1) \cap Z(f_2)$. Since $\omega_1 \neq 0$, we conclude that $E \cap Z(f_1) \cap Z(f_2)$ is not empty, and taking g in this set we get, using the fact that $f \in I(f_1,f_2) \cap [f_1,f_2]_E$:

$$\Phi(0,0) = \Phi(f_1(g),f_2(g)) = f(g) = 0.$$

Let α, β ,be constants such that $|\alpha| + |\beta| > 0$, and

$$\alpha(D_1\Phi)(0,0) + \beta(D_2\Phi)(0,0) = 0. \tag{3.9}$$

We define a bounded linear functional,

$$\omega = \alpha\omega_1 + \beta\omega_2.$$

By (3.7), (3.8), and (3.9) we have for every $h \in A(G)$

$$\langle \omega, fh \rangle = 0.$$

That is, ω annihilates $I(f)$. Since $\langle \omega, \bar{\alpha}f_2 + \bar{\beta}f_2 \rangle = (|\alpha|^2 + |\beta|^2)m_1(0,0) \neq 0$, ω does not annihilate $I(f_1,f_2)$, and the theorem is proved.

COROLLARY. *For every non-discrete locally compact abelian group G, A(G) contains a closed sub-algebra with closed ideals which are not singly generated.*

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